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Convergence of a Class of Interpolatory Splines for Holomorphic Functions

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1. INTRODUCTION

Let $\Gamma_\rho = \{z\colon |1-z^2| = \rho\}$, $\rho > 1$, denote a lemniscate containing $[-1, 1]$ and let D_ρ denote the interior of Γ_ρ. Let $A(D_\rho)$ denote the class of functions holomorphic in the domain D_ρ and let the maximum norm on $[-1, 1]$ be denoted by $\|\cdot\|$. Suppose that $S(x) = (S_{m,n}f)(x)$ denotes the complete spline interpolant of degree $2m-1$ to f on $[-1, 1]$ with respect to any given knots $y_1 < y_2 < \cdots < y_n$ in $(-1, 1)$. More precisely, $S(x) \in C^{2m-2}[-1, 1]$, and

$$S(y_\nu) = f(y_\nu), \qquad \nu = 1, 2, ..., n, \tag{1.1}$$

$$S^{(j)}(\pm 1) = f^{(j)}(\pm 1), \qquad j = 0, 1, ..., m-1. \tag{1.2}$$

In connection with Professor Schoenberg's 1968 conjecture regarding the convergence of the spline interpolant $(S_{m,n}f)(x)$ as $m \to \infty$, it was recently proved [2] that if $f \in A(D_\rho)$, $\rho > 1$, then the spline interpolant converges to f geometrically on $[-1, 1]$.

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The purpose of this note is to examine the sensitivity of the above results to different types of interpolatory conditions. A close examination of the proof of the result in [2] shows that their result prevails even when their condition is replaced by

$$S(x_\nu) = f(x_\nu), \qquad \nu = 1, 2, ..., n,$$

when $-1 < x_1 < \cdots < x_n < 1$ are any n distinct points, not necessarily the same as the knots $\{y_j\}_1^n$.

In view of this, we propose to consider the following

PROBLEM. Let $m \geqslant 1$, $d \geqslant 0$ be integers and let $\{y_j\}_1^n$,

$$-1 < y_1 < y_2 < \cdots < y_n < 1 \tag{1.3}$$

be n knots. Let $\mathscr{S}_M = \mathscr{S}_M(y_1, ..., y_2)$ denote the class of splines of degree $M := 2m + d - 1$ with knots $\{y_j\}_1^n$ each taken with multiplicity $\{\beta_j\}_1^n$, $\beta_j \geqslant 1$. For a given positive integer l, let

$$-1 < x_1 < x_2 < \cdots < x_l < 1 \tag{1.4}$$

be given, where each x_i is taken with multiplicity α_i, $\alpha_i \geqslant 1$ $(i = 1, ..., l)$. We require that

$$\sum_{i=1}^{l} \alpha_i = \sum_{i=1}^{n} \beta_i. \tag{1.5}$$

We consider the interpolation problem of finding $S(x) \in \mathscr{S}_M$ satisfying

$$S^{(\nu)}(x_i) = f^{(\nu)}(x_i), \qquad \nu = 0, 1, ..., \alpha_i - 1;\ i = 1, ..., l. \tag{1.6}$$

and

$$\begin{aligned} S^{(\nu)}(1) &= f^{(\nu)}(1) \qquad && \nu = 0, 1, ..., m + d - 1, \\ S^{(\nu)}(-1) &= f^{(\nu)}(-1), \qquad && \nu = 0, 1, ..., m - 1. \end{aligned} \tag{1.7}$$

The problem is to find the convergence properties of $S(x) \in \mathscr{S}_M$, $M = 2m + d - 1$, when $m \to \infty$.

Remark. For large enough values of m, the interlacing conditions of Schoenberg and Whitney are satisfied and so the uniqueness and existence of $S(x) \in \mathscr{S}_M$ satisfying (1.6) and (1.7) follows.

We shall prove

THEOREM 1. *Let $f \in A(D_\rho)$, $\rho > 1$ and let $S_m(x) \in \mathscr{S}_M(y_1 \cdots y_n)$ satisfy* (1.6) *and* (1.7) , *where $M = 2m + d - 1$. Then*

$$\lim_{m \to \infty} \|f - S_m f\|^{1/m} \leqslant 1/\rho.$$

An additional, more precise statement will be given in Section 4.

2. PRELIMINARIES

In order to examine the above problems, we shall recall some of the notations and results of [2]. There, the method was based on the boundary conditions. The main tool in that study was the B-spline

$$K_m(x, y) = M(x \mid (-1)^m, y, 1^m) \tag{2.1}$$

of degree $2m - 1$ with a simple knot at y and m-fold knots at ± 1, normalized to have integral one. We shall need the following recursion formula for $K_m(x, y)$:

$$K_m(x, y) = \frac{m}{m-1} \left\{ (m-1) \binom{2m-2}{m-1} \left(\frac{1-x^2}{4} \right)^{m-1} \left(\frac{1-xy}{1-y^2} \right) - \frac{(x-y)^2}{(1-y^2)} K_{m-1}(x, y) \right\}. \tag{2.2}$$

In particular,

$$K_m(x) := K_m(x, x) = m \binom{2m-1}{m-1} \left(\frac{1-x^2}{4} \right)^{m-1} \tag{2.3}$$

Following [2], we set

$$T_m(x, y) = K_m(x, y)/(1 - x^2)\, K_m(x, x) \tag{2.4}$$

and

$$T_\infty(x, y) = \frac{1}{1 - xy}. \tag{2.5}$$

We shall prove the following

LEMMA 1. *We have the following identity*

$$T_\infty(x, y) - T_m(x, y)$$
$$= \frac{1}{2m(1-xy)}\left[\frac{(x-y)^2}{(1-xy)^2} + 6\rho(1+4\rho)^{1/2}\int_0^1 v^m(1+4\rho v)^{-5/2}dv\right] \tag{2.6}$$

where

$$\rho = \frac{(1-x^2)(1-y^2)}{4(x-y)^2}. \tag{2.7}$$

Proof. The proof follows from the known relation [2],

$$T_m(x, y) = m\left(1-\frac{1}{2m}\right)T_\infty(x, y)(1+4\rho)^{1/2}\int_0^1 v^{m-1}(1+4\rho v)^{-1/2}dv,$$

on writing

$$\begin{aligned} &T_\infty(x, y) - T_m(x, y)\\ &= mT_\infty(x, y)\int_0^1 v^{m-1}\left[\left(1-\frac{1}{2m}\right)(1+4\rho)^{1/2}(1+4\rho v)^{-1/2}\right]dv\\ &= mT_\infty(x, y)\left[\int_0^1 v^{m-1}\{1-(1+4\rho)^{1/2}(1+4\rho v)^{-1/2}\}\,dv\right.\\ &= mT_\infty(x, y)\left[\int_0^1 v^{m-1}\{1-(1+4\rho)^{1/2}(1+4\rho v)^{-1/2}\}\,dv\right.\\ &\quad\left.+\frac{1}{2m}\int_0^1 v^{m-1}(1+4\rho)^{1/2}(1+4\rho v)^{-1/2}dv\right]. \end{aligned} \tag{2.8}$$

Using integration by parts on the first integral and simplifying, we get (2.6).

Remark 2. Formula (2.6) shows that $T_\infty(x, y) \geqslant T_m(x, y)$ and that

$$\frac{1}{1-xy} - T_m(x, y) = \frac{1}{2m}\frac{(x-y)^2}{(1-xy)^3} + O(m^{-2}). \tag{2.9}$$

From (2.6) and (2.7) we see that (2.9) is uniformly valid for any compact set in $[0, 1) \times [0, 1)$.

More generally, for a fixed integer $d \geqslant 0$, we set

$$K_{m+d,m}(x, y) := M(x \mid (-1)^{m+d}, y, 1^m). \tag{2.10}$$

and in analogy with (2.4), set

$$T_{m+d,m}(x, y) := \frac{K_{m+d,m}(x, y)}{(1-x^2)\, K_{m+d,m}(x, x)}. \tag{2.11}$$

We shall prove the recursion

LEMMA 2. *If $d \geqslant 0$, then we have*

$$\begin{aligned} K_{m+d,m}(x, y) &= \frac{2m+d}{2m}\left(\frac{y-x}{1+y}\right)^d K_m(x, y) \\ &\quad + \left(\frac{1-x^2}{4}\right)^m \left(\frac{1-x}{2}\right)^d \frac{2m+d}{y-x} \\ &\quad \times \sum_{j=1}^{d} \binom{2m+d-j-1}{m+d-j} \left\{\frac{2(j-x)}{(1+y)(1-x)}\right\}^j. \end{aligned} \tag{2.12}$$

Proof. We begin with the known recursion formula for B-splines and see that if $p > m$, we have

$$K_{p,m}(x, y) = \frac{m+p}{m+p-1}\left\{\frac{y-x}{1+y} K_{p-1,m}(x, y) + \frac{x+1}{1+y} M(x \mid (-1)^p, 1^p)\right\}.$$

Since

$$M(x \mid (-1)^p, 1^m) = p\binom{p+m-1}{m-1}\frac{(1+x)^{m-1}(1-x)^{p-1}}{2^{m+p-1}},$$

we get for any integer s, $1 \leqslant s \leqslant p-m$,

$$\begin{aligned} K_{p,m}(x, y) &= \frac{p+m}{p+m-s}\left(\frac{y-x}{1+y}\right)^s K_{p-s,m}(x, y) \\ &\quad + \left(\frac{1-x}{2}\right)^p \left(\frac{1+x}{2}\right)^m \frac{p+m}{y-x} \\ &\quad \times \sum_{j=1}^{s} \binom{p+m-j-1}{p-j} \left\{\frac{2(y-x)}{(1+y)(1-x)}\right\}^j. \end{aligned}$$

Formula (2.12) follows from the above when $p = m+d$ and $s = d$.

We are interested in expressing $T_{m+d,m}(x, y)$ in terms of $T_\infty(x, y)$ as in Lemma 1. This is done in

LEMMA 3. *We have*

$$T_\infty(x, y) - T_{m+d,m}(x, y)$$
$$= \frac{1}{2m} \left\{ \frac{y-x}{(1+y)(1-x)} \right\}^d \frac{(x-y)^2}{(1-xy)^3} + O(m^{-2}). \tag{2.13}$$

Proof. From (2.12), putting $x = y$, we obtain

$$K_{m+d,m}(x, x) = \frac{1}{2}\left(\frac{1-x^2}{4}\right)^{m-1}\left(\frac{1-x}{2}\right)^d (2m+d)\binom{2m+d-2}{m+d-1} \tag{2.14}$$

and using (2.3), we also have

$$K_{m+d,m}(x, x)$$
$$= K_m(x)\cdot\left(\frac{1-x}{2}\right)^d \frac{2m+d}{2m}\binom{2m+d-2}{m+d-1}\Big/\binom{2m-2}{m-1}.$$

From the above, an easy computation shows that

$$(1-x^2)\, K_{m+d,m}(x, x) = (1-x^2)\, K_m(x)(1-x)^d\, (1 + O(1/m)). \tag{2.15}$$

Since for $1 \leqslant j \leqslant d$, we have

$$\binom{2m+d-j-1}{m+d-j}\Big/\binom{2m+d-2}{m+d-1} = \frac{1}{2^{j-1}}\left(1 + O\left(\frac{1}{m}\right)\right), \tag{2.16}$$

it follows from (2.12) on dividing both sides by $(1-x^2)\, K_{m+d,m}(x, x)$ and on using (2.14), (2.15), and (2.16) that

$$T_{m+d,m}(x, y)$$
$$= \left\{\frac{y-x}{(1+y)(1-x)}\right\}^d T_m(x, y)\left(1 + O\left(\frac{1}{m}\right)\right)$$
$$+ \frac{1}{y-x}\sum_{j=1}^{d}\left\{\frac{y-x}{(1+y)(1-x)}\right\}^j\left(1 + O\left(\frac{1}{m}\right)\right)$$
$$= \left[\left\{\frac{y-x}{(1+y)(1-x)}\right\}^d T_m(x, y) + \frac{1 - \{(y-x)/(1+y)(1-x)\}^d}{1-xy}\right]$$
$$\times\left(1 + O\left(\frac{1}{m}\right)\right)$$
$$= \left[\frac{1}{1-xy} + \left\{\frac{y-x}{(1+y)(1-x)}\right\}^d (T_m(x, y) - T_\infty(x, y))\right]\left(1 + O\left(\frac{1}{m}\right)\right)$$

which gives (2.13) on recalling (2.9).

3. Interpolation within the Subspace $\mathscr{S}_m^0$.

We denote by $\mathscr{S}_M^0 = \mathscr{S}_M^0(y_1,..., y_n)$ the subspace of $\mathscr{S}_M(y,..., y_n)$, where $S(x) \in \mathscr{S}_M^0$ satisfies

$$\begin{aligned} S^{(\nu)}(1) &= 0, \qquad \nu = 0, 1,..., m+d-1, \\ S^{(\nu)}(-1) &= 0, \qquad \nu = 0, 1,..., m-1. \end{aligned} \tag{3.1}$$

Clearly, $\dim \mathscr{S}_M^0 = \sum_{i=1}^n \beta_i$. From (2.10) (or equivalently from (2.12)), we see that

$$\frac{\partial^\nu}{\partial y^\nu} K_{m+d}(x, y)\bigg|_{x=+1} = 0, \qquad \nu = 0, 1,..., m+d-1,$$

$$\frac{\partial^\nu}{\partial y^\nu} K_{m+d}(x, y)\bigg|_{x=-1} = 0, \qquad \nu = 0, 1,...,m-1.$$

For typographical reasons, we shall denote $\partial^\nu/\partial y^\nu$ by ∂_y^ν and $\partial^\nu/\partial x^\nu$ by ∂_x^ν. Then as a basis for the space $\mathscr{S}_M^0$, we take the functions $J_{j,\nu}(x)$ defined by

$$J_{j,\nu}(x) := \partial_{y_j}^\nu K_{m+d,m}(x, y_j), \qquad \nu = 0, 1,..., \beta_j - 1, j = 1, 2,..., n. \tag{3.2}$$

For sufficiently large m, these $\sum_1^n \beta_j$ functions are linearly independent as will be evident below.

In order to state the result of this section, we use the standard notation for a Fredholm determinant (see [1]) and define the linear operator

$$(Hf)(x) = N(x)/D \tag{3.3}$$

where D is determinant of order $\sum_1^d \alpha_i$ $(= \sum_{i=1}^n \beta_i)$ given by

$$D := T_\infty \begin{pmatrix} \{x_1\}^{\alpha_1},..., \{x_i\}^{\alpha_i} \\ \{y_1\}^{\beta_1},..., \{y_n\}^{\beta_n} \end{pmatrix}. \tag{3.4}$$

Here, $\{x_i\}^{\alpha_i}$ is an abbreviation for x_i taken α_i times, and so on. Also, $N(x)$ is a determinant obtained from D by adjoining to it a first row a and a last column b where the row a consists of the functions $\partial_{y_j}^\nu T_\infty(x, y_j)$, $\nu = 0, 1,..., \beta_j - 1$ and $j = 1, 2,..., n$ and the element 0; the column b comprises 0 and $f^{(\nu)}(x_i)$, $\nu = 0, 1,..., \alpha_i - 1$, and $i = 1, 2,..., l$.

Since $T_\infty(x, y)$ is an extended totally positive kernel, it follows that $D \neq 0$ [1]. From (3.3), it is clear that

$$(Hf)^{(\nu)}(x_i) = f^{(\nu)}(x_i), \qquad \nu = 0, 1,..., \alpha_i - 1, i = 1,..., l. \tag{3.5}$$

We shall now prove

PROPOSITION 1. *If $F_m(x)=(1-x^2)^m(1-x)^d f(x)$ for f sufficiently differentiable and if $S_m(F_m)\in\mathscr{S}_M^0$ satisfies* (1.6) *and* (1.7) (*with F_m instead of f*), *then*

$$\lim_{m\to\infty}(1-x^2)^{-m}(1-x)^{-d}S_m(F_m)(x)=(Hf)(x) \tag{3.6}$$

uniformly in $-1\leqslant x\leqslant 1$.

Proof. The interpolant S_m has the representation

$$(S_mF_m)(x)=\sum_{j=1}^{n}\sum_{\nu=0}^{\beta_j-1}d_{j\nu}^{(m)}J_{j,\nu}(x,y_j) \tag{3.7}$$

where the $d_{j\nu}^{(m)}$ are determined by the interpolatory conditions

$$(\partial_{x_i}^k S_m(F_m))(x_i)=F_m^{(k)}(x_i),\qquad k=0,\dots,\alpha_i-1;\ i=1,\dots,l. \tag{3.8}$$

Using (3.7), we see that the conditions (3.8) are equivalent to the following system of linear equations

$$\sum_{j=1}^{s}\sum_{\nu=0}^{\beta_j-1}\tilde{d}_{j\nu}^{(m)}\,\partial_{x_i}^k\,\partial_{y_j}^\nu\,T_{m+d,m}(x_i,y_j)=f^{(k)}(x_i),$$
$$k=0,1,\dots,\alpha_i-1;\quad i=1,2,\dots,l. \tag{3.9}$$

This is easily checked on using (2.11) since both F_m and $S_m(F_m)$ satisfy

$$F_m^{(\nu)}(1)=(S_mF_m)^{(\nu)}(1)=0,\quad \nu=0,1,\dots,m+d-1,$$

and

$$F_m^{(\nu)}(-1)=(S_mF_m)^{(\nu)}(-1)=0,\quad \nu=0,1,\dots,m-1.$$

Dividing both sides of (3.7) by $(1-x^2)^m(1-x)^d$ and recalling (2.11), we see that

$$(1-x^2)^{-m}(1-x)^{-d}(S_mF_m)(x)=\sum_{j-1}^{n}\sum_{\nu=0}^{\beta_j-1}\tilde{d}_{j\nu}^{(m)}\,\partial_{y_j}^\nu\,T_{m+d,m}(x,y_j).$$

Eliminating $\tilde{d}_{j\nu}^{(m)}$ from the above and from (3.9), we obtain

$$(1-x^2)^{-m}(1-x)^{-d}(S_mf_m)(x)=N_m(x)/D_m(x)$$

where D_m is obtained from (3.4) by using the kernel $T_{m+d,m}(x,y)$ in place of $T_\infty(x,y)$ and $N_m(x)$ is obtained from $N(x)$ likewise. The relation (3.6) now follows by letting $m\to\infty$ and using Lemma 3.

Remark. In case $l=n$ and $\alpha_i=\beta_i=2$, $i=1,..., n$, an explicit expression for $(Hf)(x)$ can be easily given. Indeed, set

$$B_n(x) := \prod_{j=1}^{n} \frac{x-x_j}{1-xy_j}. \tag{3.10}$$

Then

$$(Hf)(x) = \sum_{\nu=1}^{n} f(x_\nu)\, h_\nu(x) + \sum_{\nu=1}^{n} f'(x_\nu)\, H_\nu(x), \qquad f \in C'[-1, 1] \tag{3.11}$$

where

$$\begin{aligned} h_\nu(x) &= \left\{1 - \frac{B_n''(x_\nu)}{B_n'(x_\nu)}(x-x_\nu)\right\} l_\nu^2(x) \\ H_\nu(x) &= (x-x_\nu)\, l_\nu^2(x) \\ l_\nu(x) &= \frac{B_n(x)}{(x-x_\nu)\, B_n'(x_\nu)}, \qquad \nu = 1, 2,..., n. \end{aligned} \tag{3.12}$$

Here, $h_\nu(x)$ and $H_\nu(x)$ are linear combinations of $\{(1-xy_j)^{-1}\}_{j=1}^n$ and $\{(1-xy_j)^{-2}\}_{j=1}^n$ and $(Hf)(x)$ is the linear combination of these $2n$ functions which interpolates f in the Hermite sense at the nodes $\{x_\nu\}_{\nu=1}^n$.

4. Proof of Theorem 1

We can now state more completely

THEOREM 1. *Let $f \in A(D_\rho)$, $\rho > 1$ and let $S_m(x) \in \mathscr{S}_M(y_1,..., y_n)$ satisfy* (1.6) *and* (1.7), *where $M = 2m + d - 1$. Then*

$$\lim_{m \to \infty} \|f - S_m f\|^{1/m} \Leftarrow 1/\rho$$

Moreover, for any $\xi \notin [-1, 1]$, we have

$$\lim_{m \to \infty} \frac{(1-\xi^2)^m (1-\xi)^d}{(1-x^2)^m (1-x)^d} (f_\xi(x) - (S_m f_\xi)(x)) = f_\xi(x) - Hf_\xi(x)$$

uniformly for $x \in [-1, 1]$, where $f_\xi(x) = 1/(\xi - x)$.

Proof. Let $(P_M f)(x)$ denote the polynomial of degree $M = 2m + d - 1$

which interpolates f and its first $m+d-1$ derivatives at 1, while at -1 it interpolates f and its first $m-1$ derivatives. Then

$$f(x)-(P_M f)(x)=\frac{1}{2\pi i}\int_\Gamma \frac{(1-x^2)^m(1-x)^d f(z)}{(z-x)(1-z^2)^m(1-z)^d}\,dz \tag{4.1}$$

where Γ is any contour containing the interior of Γ_1. If $\xi \notin [-1, 1]$ then for $f_\xi(x)$ we have

$$f_\xi(x)-(P_M f_\xi)(x)=\frac{(1-x^2)^m(1-x)^d}{(\xi-x)(1-\xi^2)^m(1-\xi)^d} \tag{4.2}$$

and

$$\begin{aligned} &f_\xi(x)-S_m f_\xi(x)\\ &\quad=f_\xi(x)-(P_M f_\xi)(x)-\frac{1}{(1-\xi^2)^m(1-\xi)^d}(S_m F_{m,\xi})(x) \end{aligned} \tag{4.3}$$

where

$$F_{m,\xi}(x)=(1-x^2)^m(1-x)^d/(\xi-x).$$

Using Proposition 1, we have

$$\lim_{m\to\infty}\frac{(1-\xi^2)^m(1-\xi)^d}{(1-x^2)^m(1-x)^d}(f_\xi(x)-(S_m f_\xi)(x))=f_\xi-Hf_\xi(x).$$

To prove the first part of the theorem, suppose $f\in A(D_\rho)$ for some $\rho>1$. In (4.1), use the contour Γ_ρ and conclude

$$f(x)-(P_M f)(x)=O(\rho^{-m}) \tag{4.4}$$

uniformly for $-1\leqslant x\leqslant 1$. Then

$$\begin{aligned} &f(x)-(S_m f)(x)\\ &\quad=f(x)-(P_m f)(x)+(S_m(f-P_M))(x)\\ &\quad=f(x)-(P_M f)(x)-\frac{1}{2\pi i}\int_{\Gamma_\rho}\frac{f(z)}{(1-z^2)^m(1-z)^d}(S_m F_{m,z})(x)\,dz \end{aligned}$$

where

$$F_{m,z}(x)=(1-x^2)^m(1-x)^d/(z-x).$$

By (3.6), $(S_m F_m, z)(x)$ is clearly uniformly bounded only for $x\in[-1, 1]$ and $z\in\Gamma_\rho$, $\rho>1$. This together with (4.4) gives the desired result.

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